SWIM Princeton 2009

Ford Circles

In this project, we made connections between Ford circles and the Apollonian Circle Packing (0, 0, 1, 1).

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We will refer to a circle with curvature x as circle x.

We made observations about the curvatures of the ACP Circles.

Claim: If circle n^2 is tangent to circle 1 and circle 0, (the line) then the two circles tangent to circle n^2, circle 0, and circle 1 have curvature $(n-1)^2$ and $(n+1)^2$.

(If d = the curvature of the two unknown circles)

Proof: From Descartes' theorem, we know that:

$$1^2 + (n^2)^2 + 0^2 + d^2 = \frac{1}{2}(1 + n^2 + 0 + d)^2$$

$$1 + n^4 + 0 + d^2 = \frac{1}{2}(1 + n^2 + d)^2$$

$$1 + n^4 + d^2 = \frac{1}{2}(1 + n^2 + d)(1 + n^2 + d)$$

$$2 + 2n^4 + 2d^2 = 1 + n^2 + d + n^2 + n^4 + dn^2 + d + dn^2 + d^2$$

$$2 + 2n^4 + 2d^2 = 1 + 2n^2 + 2d + n^4 + 2dn^2 + d^2$$

$$1 + n^4 + d^2 = 2n^2 + 2d + 2dn^2$$

$$d^2 - 2d - 2dn^2 + 1 + n^4 - 2n^2 = 0$$

$$d^2 + d(-2 - 2n^2) + (1 + n^4 - 2n^2) = 0$$

Then, using the quadratic formula, we know that:

$$d = \frac{2 + 2n^2 \pm \sqrt{(-2-2n^2)^2 - 4(1)(1+n^4-2n^2)}}{2(1)}$$

$$d = \frac{2 + 2n^2 \pm \sqrt{4 + 8n^2 + 4n^4 - 4 - 4n^4 + 8n^2}}{2}$$

$$d = \frac{2 + 2n^2 \pm \sqrt{16n^2}}{2}$$

$$d = \frac{2 + 2n^2 \pm 4n}{2}$$

$$d = 1 + n^2 \pm 2n$$

$$d = 1 + n^2 + 2n \quad \text{or} \quad d = 1 + n^2 - 2n$$

$$d = (n+1)^2 \quad \text{or} \quad d = (n-1)^2$$

We also made the following observation:

Claim: If circles n^2 and $(n+1)^2$ are tangent to circle 0, the two circles tangent to the circle 0, circle n^2 and circle $(n+1)^2$ have curvatures 1 and $(2n+1)^2$, respectively.

(d = the curvature of the two unknown circles)

Proof:
$$0^2 + (n^2)^2 + [(n+1)^2]^2 + d^2 = \tfrac{1}{2}(n^2 + (n+1)^2 + 0 + d)^2$$
$$n^4 + (n+1)^4 + d^2 = \tfrac{1}{2}(n^2+(n+1)^2+d)^2$$
$$2n^4 + 2(n+1)^4 + 2d^2 = (n^2+(n+1)^2+d)^2$$
$$\cancel{2}n^4 + \cancel{2}(n+1)^4 + \cancel{2}d^2 = n^4 + (n+1)^4 + d^2 + 2n^2(n+1)^2 + 2n^2d + 2(n+1)^2d$$
$$n^4 + (n+1)^4 + d^2 = 2n^2(n+1)^2 + 2n^2d + 2(n+1)^2d$$
$$d^2 - 2n^2d - 2(n+1)^2d + n^4 + (n+1)^4 - 2n^2(n+1)^2 = 0$$
$$d^2 + d(-2n^2 - 2(n+1)^2) + [n^4 + (n+1)^4 - 2n^2(n+1)^2] = 0$$

$$d = \frac{2n^2 + 2(n+1)^2 \pm \sqrt{(-2n^2 - 2(n+1)^2)^2 - 4(1)(n^4 + (n+1)^4 - 2n^2(n+1)^2)}}{2}$$

$$d = \frac{2n^2 + 2n^2 + 4n + 2 \pm \sqrt{(-2n^2 - 2n^2 - 4n - 2)^2 - 4(n^4 + n^4 + 4n^3 + 6n^2 + 4n + 1 - 2n^2(n^2+2n+1))}}{2}$$

$$d = \frac{4n^2 + 4n + 2 \pm \sqrt{(-4n^2 - 4n - 2)^2 - 4(\cancel{2n^4} + \cancel{4n^3} + 6n^2 + 4n + 1 - \cancel{2n^4} - \cancel{4n^3}}}{2}$$

$16n^4 + 16n^2 + 4$

$32n^3 + 16n^2 + 16n$

$$d = \frac{4n^2 + 4n + 2 \pm \sqrt{16n^4 + 32n^3 + 32n^2 + 16n + 4 - 4(4n^2 + 4n + 1)}}{2}$$

$$d = \frac{4n^2 + 4n + 2 \pm \sqrt{16n^4 + 32n^3 + 32n^2 + \cancel{16n} + \cancel{4} - 16n^2 - \cancel{16n}}}{2}$$

$$d = \frac{4n^2 + 4n + 2 \pm \sqrt{16n^4 + 32n^3 + 16n^2}}{2}$$

$$d = \frac{4n^2 + 4n + 2 \pm \sqrt{16n^2(n^2 + 2n + 1)}}{2}$$

$$d = \left[4n^2 + 4n + 2 \pm \sqrt{(4n)^2(n+1)^2}\right]/2$$
$$d = [4n^2 + 4n + 2 \pm 4n(n+1)]/2$$
$$d = 2n^2 + 2n + 1 \pm 2n(n+1)$$
$$d = 2n^2 + 2n + 1 \pm 2n^2 + 2n$$

continued to the left

$d = 2n^2 + 2n + 1 + 2n^2 + 2n,$

$= 2n^2 + 2n + 1 - 2n^2 - 2n$

$d = 4n^2 + 4n + 1,$

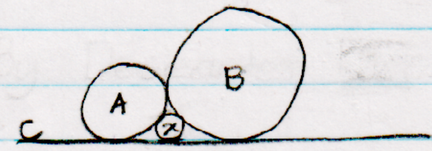
$= 1$

$d = (2n+1)^2$

$= 1$

<u>Claim</u>: All ACP circles tangent to one of the parallel lines have a square curvature.

<u>Proof</u>:



Let A, B, C, and x be curvatures of four mutually tangent circles, where A and B are known to be squares. Then by Descartes' Theorem, we know that

$$\tfrac{1}{2}(A+B+C+x)^2 = A^2 + B^2 + C^2 + x^2$$

Since one of the circles is a line, we know that one of the curvatures is 0, so

$$\tfrac{1}{2}(A+B+0+x)^2 = A^2 + B^2 + 0^2 + x^2$$

$$\tfrac{1}{2}(A+B+x)^2 = A^2 + B^2 + x^2$$

Expanding and simplifying, we get

$$A^2+B^2+C^2+2AB+2AC+2BC = 2A^2+2B^2+2x^2$$

$$A^2+B^2+x^2-2Ax-2Bx-2AB = 0$$

$$x^2-2(A+B)x+(A-B)^2 = 0$$

By quadratic formula

$$x = \frac{2(A+B) \pm \sqrt{(-2A+B)^2 - 4(1)(A-B)^2}}{2(1)}$$

Simplifying,

$$x = \frac{2(A+B) \pm \sqrt{4((A+B)^2 - (A-B)^2)}}{2}$$

$$= (A+B) \pm \frac{2\sqrt{A^2+2AB+B^2-A^2+2AB-B^2}}{2}$$

$$= A+B \pm \sqrt{4AB}$$

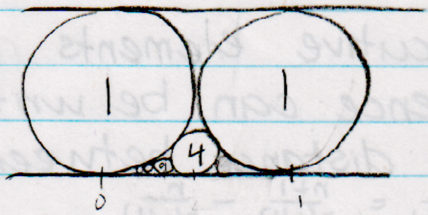
$$= A+B \pm 2\sqrt{AB}$$

$$= A \pm 2\sqrt{AB} + B$$

$$= (\sqrt{A} \pm \sqrt{B})^2$$

Since we know that A and B are squares, $\sqrt{A} \pm \sqrt{B}$ is an integer, and x is also a perfect square. We know that we can start with squares because 1 and 4 are curvatures of circles

tangent to each other and the line.
We can now generalize from this that all circles tangent to the line have square curvatures. ■



Claim: If you label the center of a circle with curvature 1 as 0 on the number line, and the circle of curvature 1 tangent to it on the right as 1, then the centers of all the circles tangent to the line and the left circle with curvature 1 form the harmonic sequence.

Proof: Let B and C be radii of two tangent circles in the set of circles described above. We draw a right triangle using the segment connecting the centers of the two circles as the hypotenuse. By Pythagorean Theorem:

$$(B-C)^2 + x^2 = (B+C)^2$$

Simplifying.

$$x^2 = (B+C)^2 - (B-C)^2$$
$$x^2 = B^2 + 2BC + C^2 - B^2 + 2BC - C^2$$
$$x^2 = 4BC$$
$$x = 2\sqrt{BC}$$

Since we labelled the distance between the circles of curvature 1 as 1 when it should be 2, the actual distance is $x = \sqrt{BC}$.

We showed earlier that this "string" contains circles with consecutive square curvatures. Knowing this, we can rewrite the radii as $B = \frac{1}{m^2}$ and $C = \frac{1}{(m+1)^2}$, since the curvature is the reciprocal of the radius.

Therefore, $x = \sqrt{\left(\frac{1}{m^2}\right)\left(\frac{1}{(m+1)^2}\right)}$

$$x = \frac{1}{m(m+1)}$$

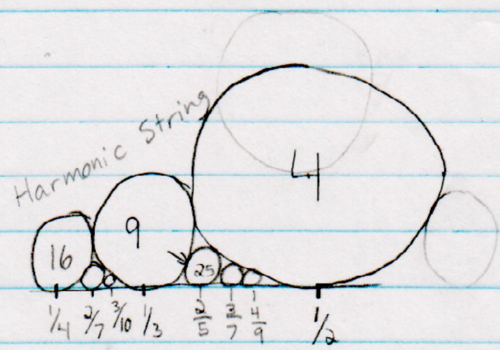
Two consecutive elements in the harmonic sequence can be written as $\frac{1}{n}$ and $\frac{1}{n+1}$. The distance between them is then $\frac{1}{n} - \frac{1}{n+1} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)}$
$= \frac{1}{n(n+1)}$

This is equivalent to x, the distance between the centers of two tangent circles in this string. The circle with curvature 4 has a center that lies at $\frac{1}{2}$, which is the first number in the harmonic sequence. Therefore, all circles in this string have centers that fall on the harmonic sequence. We name this the harmonic string.

Furthermore, because we know that this string consists of both centers on the harmonic sequence and circles with curvatures of consecutive squares, we can see by the fact that the circle with curvature 4's center lies at $\frac{1}{2}$ that the center of each circle in the harmonic string lies at the reciprocal of the square root of its curvature on our number line. :)

Patterns in the Centers

We observed some patterns in the x-coordinates of the centers of circles in strings coming off the Harmonic String.



Starting with $\frac{1}{3}$ (circle 9) and following the string of circles tangent to circle 4 and the line, the following x-coordinates occur:

$\frac{1}{3}$ $\frac{2}{5}$ $\frac{3}{7}$ $\frac{4}{9}$

This appears to be the pattern: $\frac{n}{2n+1}$

Similarly, starting with $\frac{1}{4}$ (circle 16) and following the string of circles tangent to circle 9 and the line, the following x-coordinates occur:

$\frac{1}{4}$ $\frac{2}{7}$ $\frac{3}{10}$

This appears to be the pattern: $\frac{n}{3n+1}$

Conjecture: If x is the square root of the curvature of a particular harmonic circle, the centers of the string of circles coming off that harmonic circle follow this pattern:

$$\frac{n}{(x-1)n+1}$$

Introduction to Ford Circles

Definition: A Ford Circle is defined as $C(\frac{p}{q})$ with p and q positive integers. For a Ford Circle $C(\frac{p}{q})$, the center $= (\frac{p}{q} + \frac{1}{2q^2} i)$ and the radius $= \frac{1}{2q^2}$

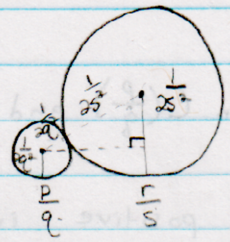
By definition, every rational number on the interval $[0,1]$ can be represented by a Ford circle.

The Ford circles have properties relating to tangency such that no two Ford circles overlap, as seen by the following cases.

We made the following observations about the tangencies of Ford Circles.

Claim: If 2 circles, $C(\frac{p}{q})$ and $C(\frac{r}{s})$, are tangent, the following relationship is true:

$$|qr - ps| = 1$$



Proof: Because if they are tangent:

$$\left(\frac{1}{2s^2} - \frac{1}{2q^2}\right)^2 + \left(\frac{r}{s} - \frac{p}{q}\right)^2 = \left(\frac{1}{2q^2} + \frac{1}{2s^2}\right)^2$$

$$\left(\frac{1}{4}\right)\left(\frac{1}{s^4} - 2\left(\frac{1}{s^2}\right)\left(\frac{1}{q^2}\right) + \frac{1}{q^4}\right) + \frac{r^2}{s^2} - 2\left(\frac{p}{q}\right)\left(\frac{r}{s}\right) + \frac{p^2}{q^2} = \frac{1}{4}\left(\frac{1}{q^4} + 2\left(\frac{1}{q^2}\right)\left(\frac{1}{s^2}\right) + \frac{1}{s^4}\right)$$

$$\frac{1}{s^4} - \frac{2}{s^2q^2} + \frac{1}{q^4} + \frac{4r^2}{s^2} - 8\frac{pr}{qs} + \frac{4p^2}{q^2} = \frac{1}{q^4} + \frac{2}{q^2s^2} + \frac{1}{s^4}$$

$$\frac{4r^2}{s^2} + \frac{4p^2}{q^2} - \frac{4}{s^2q^2} - \frac{8pr}{qs} = 0$$

$$\frac{r^2}{s^2} - \frac{2pr}{qs} + \frac{p^2}{q^2} = \frac{1}{s^2q^2}$$

$$r^2q^2 - 2pqrs + p^2s^2 = 1$$

$$(rq - ps)^2 = 1$$

$$|rq - ps| = 1$$

Claim If $|rq - ps| < 1$, then $C(\frac{p}{q})$ and $C(\frac{r}{s})$ are the same circle.

Proof Because p, q, r, and s are integers, $rq - ps$ is also an integer. Further, $|rq - ps|$ is a nonnegative integer. The only non-negative integer less than 1 is 0, so

$$|qr - ps| = 0$$

$$qr = ps$$

see back page

$$\frac{qr}{qs} = \frac{ps}{sq}$$

$$\boxed{\frac{r}{s} = \frac{p}{q}}$$

Claim: If $|rq - ps| > 1$, then $C(\frac{p}{q})$ and $C(\frac{r}{s})$ do not intersect.

Proof: Note: p, q, r and s are positive integers

$$|rq - ps| > 1$$

$$(rq - ps)^2 > 1^2$$

$$\frac{r^2q^2 - 2pqrs + p^2s^2}{s^2q^2} > \frac{1}{s^2q^2}$$

$$\frac{r^2}{s^2} - \frac{2pr}{sq} + \frac{p^2}{q^2} > \frac{1}{s^2q^2}$$

$$\frac{4r^2}{s^2} - \frac{8pr}{sq} + \frac{4p^2}{q^2} - \frac{4}{s^2q^2} > 0$$

$$\left(\frac{1}{s^4} + \frac{2}{s^2q^2} + \frac{1}{q^4}\right) + \frac{4r^2}{s^2} - \frac{8pr}{sq} + \frac{4p^2}{q^2} - \frac{4}{s^2q^2} > \left(\frac{1}{s^4} + \frac{2}{s^2q^2} + \frac{1}{q^4}\right)$$

$$\frac{1}{4}\left(\frac{1}{s^4} + \frac{2}{s^2q^2} + \frac{1}{q^4} + \frac{4r^2}{s^2} - \frac{8pr}{sq} + \frac{4p^2}{q^2} - \frac{4}{s^2q^2}\right) > \frac{1}{4}\left(\frac{1}{s^4} + \frac{2}{s^2q^2} + \frac{1}{q^4}\right)$$

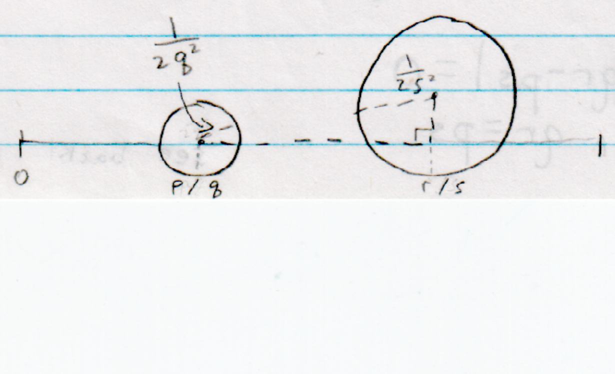
$$\frac{1}{4s^4} - \frac{1}{2s^2q^2} + \frac{1}{4q^4} + \frac{r^2}{s^2} - \frac{2pr}{sq} + \frac{p^2}{q^2} > \frac{1}{4}\left(\frac{1}{s^4} + \frac{2}{s^2q^2} + \frac{1}{q^4}\right)$$

$$\frac{1}{4}\left(\frac{1}{s^4} - \frac{2}{s^2q^2} + \frac{1}{q^4}\right) + \left(\frac{r^2}{s^2} - \frac{2pr}{sq} + \frac{p^2}{q^2}\right) > \frac{1}{4}\left(\frac{1}{s^4} + \frac{2}{s^2q^2} + \frac{1}{q^4}\right)$$

$$\left(\frac{1}{2}\left(\frac{1}{s^2} - \frac{1}{q^2}\right)\right)^2 + \left(\frac{r}{s} - \frac{p}{q}\right)^2 > \left(\frac{1}{2}\left(\frac{1}{s^2} + \frac{1}{q^2}\right)\right)^2$$

$$\left(\frac{1}{2s^2} - \frac{1}{2q^2}\right)^2 + \left(\frac{r}{s} - \frac{p}{q}\right)^2 > \left(\frac{1}{2s^2} + \frac{1}{2q^2}\right)^2$$

The following image represents what would happen for $C(\frac{p}{q})$ and $C(\frac{r}{s})$.



Claim: Ford Circles form the original ACP (0,0,2,2).

Proof: From above, two Ford circles will be tangent if and only if $|rq - ps| = 1$. We can choose 3 Ford circles in which each pair satisfies the above equation. These 3 circles will be tangent with each other because they satisfy $|rq-ps|=1$. These 3 tangent circles, along with the number line on which they were drawn, can be 4 circles within an ACP. We can apply the ACP reduction algorithm to find the root quadruple.

		$\lvert rq-ps\rvert=1$	tangent?
$\frac{p}{q} = \frac{1}{3}$	$\frac{p}{q} + \frac{r}{s}$	$\lvert(2)(3)-(1)(7)\rvert = 1$	✓
$\frac{r}{s} = \frac{2}{7}$	$\frac{p}{q} + \frac{a}{b}$	$\lvert(3)(3)-(1)(10)\rvert = 1$	✓
$\frac{a}{b} = \frac{3}{10}$	$\frac{r}{s} + \frac{a}{b}$	$\lvert(3)(7)-(2)(10)\rvert = 1$	✓

The curvatures of these 3 circles are $2q^2$, $2s^2$, and $2b^2$, based on the definition of Ford circles. The line on which these circles lie has curvature 0. Our first quadruple is:

$$\begin{bmatrix}0\\18\\98\\200\end{bmatrix} \xrightarrow{S_4} \begin{bmatrix}0\\18\\32\\98\end{bmatrix} \xrightarrow{S_4} \begin{bmatrix}0\\2\\18\\32\end{bmatrix} \xrightarrow{S_4} \begin{bmatrix}0\\2\\8\\18\end{bmatrix} \xrightarrow{S_4} \begin{bmatrix}0\\2\\2\\8\end{bmatrix} \rightarrow \begin{bmatrix}0\\0\\2\\2\end{bmatrix}$$

$\hookrightarrow -200 + 2(98+18) = 32$

$\hookrightarrow -98 + 2(32+18) = 2$

$\hookrightarrow -32 + 2(18+2) = 8$

$\hookrightarrow -18 + 2(8+2) = 2$

$\hookrightarrow -8 + 2(2+2) = 0$

Conclusion:

The ACP (0,0,2,2) is a multiple of the original circle packing (0,0,1,1). Since these ACPs represent the Ford Circles and since a Ford Circle exists for each rational number, the ACP (0,0,1,1) has a circle tangent to the line for each rational number.